Math 245C Lecture 7 Notes

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April 15, 2019

1 Bounds on Kernel Operators

1.1 Strengthening of a previous theorem

We will prove a stronger version of the following theorem.

Theorem 1.1. *Let $(X, \mathcal{M}, \mu)$ and $(\mathcal{Y}, \mathcal{N}, \nu)$ be σ-finite measure spaces. Let $K : X \times Y \to \mathbb{R}$ be $\mathcal{M} \otimes \mathcal{N}$-measurable, and let $\mathcal{F}$ be the set of $f : Y \to \mathbb{R}$ measurable functions such that $K(x, \cdot)f \in L^1$ for μ-a.e. $x \in X$. For $f \in \mathcal{F}$, define*

$$Tf(x) = \int_Y K(x, y)f(y)\, d\nu(y).$$

Assume there exists $C > 0$ such that

$$\int_Y |K(x, y)|\, d\nu(y) \leq C$$

for μ-a.e. $x \in X$ and

$$\int_X |K(x, y)|\, d\mu(x)$$

for ν-a.e. $y \in Y$. The the following conclusions hold:

1. *For any $1 \leq p < \infty$, $L^p(\nu) \subseteq \mathcal{F}$.*
2. *There exists C_p such that $\|Tf\| \leq C_p \|f\|_p$ if $f \in L^p(\nu)$.*

Recall that if $A > 0$, then

$$\phi_A(z) = \begin{cases} z & |z| < A \\ \frac{z}{|z|} A & |z| \geq A \end{cases}$$

is a function in $C(\mathbb{C}, \mathbb{C})$, and $\phi_A|_{\mathbb{R}} \in C(\mathbb{R}, \mathbb{R})$. Observe that

$$z - \phi_A(z) = \begin{cases} 0 & |z| < A \\ \frac{z}{|z|}(|z| - A) & |z| \geq A. \end{cases}$$

We shall use the notation

$$K_1 = K_1^A = K - \phi_A(K), \qquad K_2 = K_2^A = \phi_A(K).$$

Denote as T_i $(i = 1, 2)$ the operators associated to K_i $(i = 1, 2)$.

Theorem 1.2. *Let $1 \le p < \infty$ and $c > 0$. Assume that $[K(x, \cdot)]_q \le C$ for μ-a.e. $x \in X$ and $[K(\cdot, w)]_w \le C$ for ν-a.e. $y \in Y$.*

1. *If $1 \le p < \infty$, $L^p(\nu) \subseteq \mathcal{F}$.*
2. *If $1 < p < r < \infty$, then there exist $B_1 > 0$ and $B_p > 0$ such that $[Tf]_q \le B_1\|f\|_1$ and $\|Tf\|_r \le CB_p\|f\|_p$, which means T is weak type $(1, q)$ and strong type (p, r), provided that $1/r + 1 = 1/p + 1/q$.*

Proof. Let $f \in L^p(\nu)$; we want $f \in \mathcal{F}$. If $f = 0$, we are done. If $f \ne 0$, it suffices to show that $f/\|f\|_p \in \mathcal{F}$. So we need only show that if $\|f\|_p = 1$, then $f \in \mathcal{F}$. For the second conclusion, let $f \in L^p$. If $f = 0$, then the conclusion holds. If $f \ne 0$, then we can again reduce to the case $\|f\|_p = 1$ by passing to $f/\|f\|_p$. So it suffices to prove both parts when $\|f\|_p = 1$.

Let $f \in L^p(\nu)$ be such that $\|f\| = 1$. Let q' be the dual conjugate of q, and let p' be the dual conjugate of q. We have $1/r = 1/p + 1/q - 1 = 1/p - 1/q'$, and similarly, $1/r = -1/q' + 1/q$. Since $r > 0$, $1/p > 1/q'$, and $1/q > 1/p'$. So $q' > p$, and $p' > q$. We have

$$\alpha^q \lambda_{K(x,\cdot)}(\alpha) \le C, \qquad \alpha^q \lambda_{K(\cdot,y)}(\alpha) \le C.$$

To show that $|K(x, \cdot)f| \in L^1(\nu)$, we are going to show that $|K_i(x, \cdot)f| \in L1(\nu)$ for $i = 1, 2$. We have

$$\begin{aligned}\int_Y |K_1(x, y)|\, d\nu(y) &= \int_0^\infty \lambda_{K_1(x,\cdot)}(\alpha)\, d\alpha = \int_0^\infty \lambda_{K(x,\cdot)}(\alpha + A)\, d\alpha \\ &= \int_A^\infty \lambda_{K(x,\cdot)}(\alpha)\, d\alpha \le C \int_A^\infty \alpha^{-q}\, d\alpha = C\frac{A^{1-q}}{q-1}.\end{aligned}$$

The similar identity holds for $\int_X |K_1(x, y)|\, d\mu(x)$, so we have

$$\int_Y |K(x, y)|\, d\nu(y), \int_X |K(x, y)|\, d\mu(x) \le C\frac{A^{1-q}}{q-1}.$$

We have

$$\begin{aligned}\int_Y |K_2(x, y)|^{p'}\, d\nu(y) &= p' \int_0^\infty \lambda_{K_2(x,\cdot)}(\alpha)\alpha^{p'-1}\, d\alpha = p' \int_0^A \lambda_{K(x,\cdot)}(\alpha)\alpha^{p'-1} \\ &\le p' \int_0^A C\alpha^{p'-1-q}\, d\alpha = C\frac{p'}{p'-q}A^{p'-q}.\end{aligned}$$

By symmetry, we get that

$$\int_Y |K_2(x,y)|^{p'}\, d\nu(y), \int_X |K_2(x,y)|^{p'}\, d\mu(x) \le C\frac{p'}{p'-q}A^{p'-q}.$$

Apply Hölder's inequality to conclude that

$$\int_Y |K_2(x,y)f(x)|\, d\nu(y) \le \left(\int_Y |K_2(x,y)|^{p'}\, d\nu(y)\right)^{1/p'} \|f\|_p \le \left(C\frac{p'}{p'-q}\right)^{1/p'} A^{1-q/p'}$$

So $K_2(x,\cdot)f \in L^1(\nu)$. Using the previous theorem, we conclude that $K_1(x,\cdot)f \in L^1(\nu)$. In conclusion, $K(x,\cdot)f \in L^1(\nu)$, which implies that $L^p(\nu) \subseteq \mathcal{F}$.

Choosing an appropriate A: By our inequality,

$$\|T_2 f\| \le \left(C\frac{p'}{p'-q}\right)^{1/p'} A^{1-q/p'}.$$

Choose A such that

$$\left(C\frac{p'}{p'-q}\right)^{1/p'} A^{q/r} = \left(C\frac{p'}{p'-q}\right)^{1/p'} A^{1-q/p'} = \frac{\alpha}{2}.$$

That is, we choose

$$A = \left[\left(C\frac{p'}{p'-q}\right)^{1/p'} A^{q/r}\right]^{r/q}.$$

By assumption, $\|T_2 f\| \le \alpha/2$, and so $\lambda_{T_2 f}(\alpha/2) = 0$. $\square$

Next time, we will finish the proof.